Transitioning out of Poverty*

David Brasington[†], Mika Kato[‡], Willi Semmler[§]

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Abstract

In this paper, we emphasize the role of social environment and human capital formation in explaining the persistence of inequality. Recent research on inequality such as Durlauf (2004, 2006) and Brock and Durlauf (2006) proposes the role of social intereaction for socioeconomic outcomes. Others have stressed the role of education and human capital formation in explaining inequality (Mincer, 1958; Katz *et al.*, 1999, and Bowles and Gintis, 2002). We use a Romer (1990) type variety model to explain how the individual's decision of investing in education and building up skills differ under different social environment. We shall show that with stronger economies of agglomeration in the environment, the model may have two attractors, one interpreted as a poverty trap and the other interpreted as a take-off region. The long-run status and the income of individuals across groups are determined by open ended dynamics. As we show the aggregate inequality can be influenced by policy. We also present empirical evidence that indicates the existence of such two attractors.

Key Words: poverty trap, lock-ins, persistent inequality, human capital, environment effects

JEL Codes: C61, O15

1 Introduction

Adam Smith in *The Wealth of Nation* (1776) stated his idea about what may cause inequality in individual status:

> "The difference between the most dissimilar characters, between a philosopher and a common street porter, for example, seems to arise not so much from nature as from habit, custom, and education. When they

*We thank Ohio and Ohio school districts.

[†]Louisiana State University

[‡]Howard University, Washington, DC

[§]New School for Social Research, NYC

came into the world, and for the first six or eight years of their existence, they were, perhaps, very much alike, and neither their parents nor playfellows could perceive any remarkable difference."

Smith thus seems to argue that 1) social environment such as habit and custom and 2) education play a larger role in shaping individual's future than inborn talents do.

Recent research on inequality such as Durlauf (2004, 2006) and Brock and Durlauf (2006) emphasizes the role of social interaction, more broadly social environment, for socioeconomic outcomes. The social environment surrounding individuals can lead to a take-off of the individuals or can lead to their substantial immobility, the so-called social lock-in. A Brock-Durlauf type model, however, does not explicitly discuss the role of investment in education and human capital formation. In this paper, we shall introduce, following Adam Smith's thoughts, both social environment and education in a Romer (1990) type dynamic model and study why inequality persists in a certain social setting.

We present a model where a community faces decisions about consumption, human capital formation, and investment in the social environmental. The state of a community of a given point of time is characterized by human capital and a social environment. We mean by the social environment the general attractiveness of the community as a place for working, educating children, doing business, and social contacts, which is largely determined by the magnitude of social and private investments. Social investments usually provide the community with the basic living needs such as education, health care, public transportation, safety, sanitation, etc. while private investments add variety of jobs and services to the community.

The social environment can improve when the two types of investments are made although their function and nature are quite different; while the social investments can be undertaken by a community's decision board, private investments can not be enforced. The latter investments are mostly exogenous to the community's decision board but they are likely to be influenced by the community's current social environment. This may be so due to the so-called economies of agglomeration, i.e., as the community's environment improves, it can attract more private investments and with more private investments the community environment improves, although this feedback, of course, may slow down in later stages due to congestion, etc..

We incorporate agglomeration effects in the dynamics of social environment and show that with stronger economies of agglomeration, the model may exhibit two basins of attraction; one is interpreted as a poverty trap and the other as a take-off region, leading to a threshold separating two domains of attraction. The dynamics are open-ended in the sense that a community with the initial human and environmental resources above a certain threshold level tends to reach the upper steady state while a community with these initial resources below the threshold level tends to reach the lower steady state. The presence of such a mechanism has important implications for competition in the market and for aggregate inequality.[1]

In this context we can discuss public policies aimed at reducing poverty. We find that increasing the community's budget for investments in social environment works to reduce the size of the domain of the poverty trap. This implies that such a policy can provide the communities trapped in a poor state a better chance to move out of such a trap.[2]

The rest of the paper is organized as follows. Section II provides a brief survey on existing theories and empirics of human capital formation, the role of the social environment and inequality. Section III introduces our dynamic model that can explain a mechanism of lock-in and persistent inequality as well as that of take-off. With the help of a numerical study, we explore the global dynamics and derive the policy function. Section IV presents empirical evidence on educational lock-in. We use math proficiency data for school districts in Ohio for the period 1990-2002 and study whether the transition of educational attainment is affected by the original state of attainment. We find that the pattern of transition indicates a state-dependency and shows a threshold, above which the schools tend to move up to the higher level of attainment and below which the schools tend to remain in the low-attainment trap. Section V concludes the paper.

2 Human capital, social environment and inequality

It is often maintained that education and human capital formation are a fundamental force for economic growth and income increase in regions. As human capital accumulates, incomes eventually will rise, and poor regions are likely to transition out of poverty. This section briefly surveys the existing theories of human-capital-led growth and inequality and relates this to the new literature on the role of social environment. We also provide a review on the empirical evidence on these topics.

2.1 Theories of inequality

The oldest type of explanation relates income distribution to the distribution of the individual abilities. This, however, doesn't really explain why the highly skewed income distribution emerges from the normally distributed inborn abilities. As the above-cited statement by Adam Smith indicates, it is highly doubtful that inborn abilities play a relevant role in explaining the income inequality. Recently economists

[1] For a threshold model on inequality across countries, see Semmler and Ofori (2007).

[2] Success stories may include a number of gentrification projects in some areas in the NYC and Washington, DC that began in early 1990s, which eventually created hundreds of new condominiums, many new upscale restaurants, bars, shops, theaters, museums, galleries, and other attractions. This finally also led to lower crime rates.

and social scientists point that the acquired knowledge and skills, and the available resources and the environment are more critical determinants of disparities of income.

Mincer (1958) is a landmark work that relates investment in human capital in a direct way to income inequality. Mincer discusses an individual's decision on the life time allocation between training and work. As a result of different individual preferences, some choose the combination of shorter training and a low-income job with a longer life-time at work while others choose the combination of longer training and a high-income job with a shorter life-time at work. Therefore, the resulting income distribution, in his discussion, reflects a matter of individual taste and preference.

From the late 1960s to 80s, human-capital-led growth theory made great stride (Uzawa, 1965; Ben-Porath, 1967; and Lucas, 1988). Income inequality in this type of model arises from initial endowments of the individuals in addition to the individual ability of skill development. Examples of what characterizes individual's given endowments are the characteristics of parents and family members as well as those of the group, the community, the region, and the country to which an individual belongs. Becker and Tomes (1979), for example, put great emphasis on the role of family characteristics in human capital formation. Educational choice to improve his or her own abilities in their model is considered as a family's problem, especially a parent's problem, rather than an individual problem. Future income inequality, therefore, can emerge due to various degrees of parental altruism toward children and the parents' income stream. In this context then there will arise a strong likelihood of the inheritance of inequality (see Bowles and Gintis, 2002).

Yet, as above-mentioned, more recent research such as Durlauf (2004, 2006) and Brock and Durlauf (2006) puts forward a social-interaction theory of inequality. It broadly emphasizes the social environment effects based on the conjecture that the composition of groups to which a person belongs plays an important role for socioeconomic outcomes. This is, as Brock and Durlauf argue, because individual preference, beliefs and opportunities are strongly shaped and impacted by one's membership in a particular group (e.g., in a neighborhood, a school, a university, and a workplace). When positive interaction effects occur in a certain group or a social environment, this can give rise to better opportunities to the group members and create common or similar outcomes for the group members, but this may also cause a greater cross-sectional inequality and less social mobility, i.e., a considerable lock-in, unless a take off can take place.

2.2 Empirical literature

The formation of human capital is usually measured from the input side, for example, the educational expenditure or the years of schooling, see Greiner *et al.* (2005, Ch. 4). Another approach is output oriented. In this context then the quality of school seems most important and a readily available measure of school quality are proficiency tests. The current study also uses an output-based measure and employs high school proficiency test passage rates because they reliably seem to predict labor market productivity and incomes (Sander, 1996; Loury and Garman, 1995; Murnane, Willett and Levy 1995). Crown and Wheat (1995) find that increases in education help explain the convergence of incomes in the U.S. South to other regions, further underscoring the link between education, income growth, and income distribution.

A large literature examines the convergence of incomes across regions. Kubo (1995) presents a theoretical model showing how regional development can be uneven, stable, or a mixture of uneven or stable across regions. The empirical literature seems to support all these scenarios. Webber, White and Allen (2005) find U.S. states' incomes are generally converging, although some states are converging more quickly than others. Choi (2004), in contrast, finds little evidence of overall output convergence across the U.S., but finds some convergence between neighboring states. In a similar vein, Bishop, Formby and Thistle (1994) find divergence in incomes during the 1970s and 1980s.

Partridge (2005) specifically examines the link between income distribution and growth. Partridge allows for both short-run and long-run responses of income distribution to growth, and also allows for separate effects of the tails and middle of the distribution. After making these adjustments, Partridge finds that the middle-class share of income is positively related to long-run growth, as is overall income inequality. Ohio, for example, in 1999 has a highly even distribution of incomes: it ranks 40th out of 50 states in income inequality with a Gini coefficient of 0.492 (Lynch, 2003). At the same time, Ohio had an above-average increase in inequality between 1988 and 1999. Between these years, incomes in Ohio rose 3.3% in real terms, including 7.1% for the top quintile of households (Lynch, 2003).

A long and contentious literature investigates the determinants of student achievement, generally measured as proficiency test scores. As aforementioned, in contrast to input-based, we stress output-based measures of human capital. A convenient starting point is a review of the literature by Hanushek (1986), which suggests that student achievement is generally related to parent and peer characteristics, but not to school-specific inputs. Recent literature confirms the importance of parent and peer characteristics. Student achievement is positively related to the presence of two-parent households (Bonesronning, 2004; Brasington, 2007, 1999), parent income levels (Dee, 1998; Driscoll, Halcoussis and Svorny, 2003; Dewey, Husted, and Kenny, 2000), and parent or community education levels (Brasington, 2007, 1999; Dee, 1998; Bonesronning, 2004; Driscoll, Halcoussis and Svorny, 2003; Dewey, Husted, and Kenny, 2000). Some research finds that the percent of students switching schools depresses achievement (Dewey, Husted, and Kenny, 2000), although other studies find less consistent results (Coates, 2003; Brasington, 1999). Some of the empirical literature find what our model predicts, namely, poverty seems to lower student achievement (Figlio and Stone, 2001; Brasington, 1999; Dee, 1998; Driscoll, Halcoussis and Svorny, 2003).

A school's competitive environment may also be related to student achievement. Studies find that private school market share is positively (Dee, 1998; Driscoll, Halcoussis and Svorny, 2003), negatively (Zanzig, 1997), or unrelated (Brasington, 2007) to public school performance. Competition from other public schools may also matter. The number of public school districts in a county has been found to increase student achievement (Figlio and Stone, 2001), sometimes increase it (Brasington, 2007), and increase it up to a certain point, then decrease it (Zanzig, 1997).

Although some of the recent literature still finds little relationship between school-specific inputs like teacher education levels and student achievement (Brasington, 1999, 2007; Coates, 2003; Bonesronning, 2004), other papers find a relationship. Student achievement has been found positively related to teacher salary (Sander, 1993; Zanzig, 1997; Dewey, Husted, and Kenny, 2000; Figlio, 1999), per-pupil expenditures (Dee, 1998; Bonesronning, 2004; Dewey, Husted, and Kenny, 2000), a low pupil to teacher ratio (Sander, 1993; Dewey, Husted, and Kenny, 2000; Figlio, 1999), and sometimes to teacher experience and education levels (Dewey, Husted, and Kenny, 2000).

3 The model

Being aware of the critical role of education and social environment in economic growth and inequality, we next present a growth model that emphasizes the formation of human capital and the change in the social environment. We employ a Romer (1990) type of variety model that deals with two types of stock variables. One departure from the Romer type structure is that we introduce economies of agglomeration in the dynamics of social environment, which adds a strong nonlinearity to our model. With stronger agglomeration, the model possibly exhibits two basins of attraction and this allows us to explain the presence of persistent inequality.

3.1 Structure of the model

There are two state variables in our model, the human capital h and the social environmental capital s. The combination of the two state variables (s, h) describes the state of a community at that time. The environmental variable s here measures the general attractiveness of the community as a place for working, educating children, doing business, social contacts, etc.. The social environment can change largely due to social investments i and private investments f. The former investments provide the community with the basic living needs such as education, health care, public transportation, safety, sanitation, etc.; on the other hand, the latter investments may add variety of jobs and services to the community. The two types of investments are quite different in nature; while the social investments can be undertaken by the community's planner over time, private investments can not be enforced by it. The latter investments are rather exogenous to the community's decision board but yet likely to be influenced by the community's current social environment, i.e., as the community's environment improves, it can attract more private investments ($f'(s) >$

0) and vice versa. This effect, which is based on some crowding-in effect, is an important feature of the so-called economies of agglomeration. Taking these factors in account, the dynamics of the social environmental capital is described as

$$\dot{s} = i + f(s) - \delta s. \tag{1}$$

Note that we presume, in order to reduce the number of variables, that both social and private investments equally contribute to the the community's environment and the environmental capital depreciates at a common rate of δ. As it is unlikely that the scale of agglomeration increases with no limit, we additionally assume that it slows down after a certain point $\overline{s}$, i.e., $f'' > 0$ for $s < \overline{s}$, and $f'' < 0$ for $s > \overline{s}$.

Next, we assume that a community receives a constant amount of transfer k per unit of time (let's say a transfer from the local government to each community). The community's decision board that is concerned with the welfare of a typical household in the community allocates this given amount of transfer either to social investments i, to improve the community's social environment, or to spending on immediate public services j, to increase today's production or income. Note that social investments create a "stock" of environmental capital that can last for a while whereas spending on public services is a "flow" that only has a one-time effect on today's production or income.[3] The transfer cannot be carried over to the future and thus has to be exhausted at each time period. Then the allocation of the transfer is

$$i + j = k. \tag{2}$$

Production is a function of three inputs: unskilled labor l, various types of skilled labor, $x(\theta)$, and the amount of spending on public services, j. Let's assume that the number of unskilled labor in the community is constant over time. The parameter θ represents a type of skill development opportunity. Different types of skills and knowledge are acquired in different opportunities θs, e.g., represented by various types of schools and training programs Larger θ indicates that there are more diverse skill-development opportunities in the community. We assume that the degree of diversity is positively related to the community's social environment. It is likely that more diversified opportunities for skill and knowledge development are provided in a better environment.

Assuming that creation of skills follow the Cobb-Douglas type,

$$y = \gamma j^{\alpha} l^{\beta} \int_0^s x(\theta)^{1-\alpha-\beta} d\theta \tag{3}$$

[3]Investments i in social capital create the community's environmental stock– examples: public facilities such as schools, parks, libraries, museums, transportation, sewers, security system, medical facilities that will last for a while. Spending on public service j contributes to the community's production that doesn't last – examples: patrolling policemen, park and street services, etc.. The supply of those contribute to the community's productivity but j can change every time period. Thus, j is a factor to determine today's production.

where different type of skill labor $x(\theta)$ are assumed to have the same efficiency as in the Romer (1990) variety model for capital goods. This means that the equilibrium amount of skilled labor of each type is the same for all θs. We use the symbol $\widetilde{x}$ for the equilibrium size. Then the total amount of skilled labor used in production is $s\widetilde{x}$. We can simply call it human capital h.

$$\int_0^s x(\theta)d\theta = s\widetilde{x} \equiv h. \tag{4}$$

Plugging $\widetilde{x} = h/s$ into (3) gives[4]

$$y = \gamma j^{\alpha} s^{\alpha+\beta} l^{\beta} h^{1-\alpha-\beta}. \tag{5}$$

Notice that environment s now appears explicitly in the production function.

Finally, the creation of human capital is achieved by forgone consumption.

$$\dot{h} = \gamma j^{\alpha} s^{\alpha+\beta} l^{\beta} h^{1-\alpha-\beta} - c. \tag{6}$$

The community's decision board maximizes the typical household's welfare.

$$\max_{\{c,\, i\}} \int_0^{\infty} u(c) e^{-\rho t} dt \tag{7}$$

$$\begin{aligned} \text{subject to } \dot{s} &= i - \delta s + f(s) \\ \dot{h} &= \gamma j^{\alpha} s^{\alpha+\beta} l^{\beta} h^{1-\alpha-\beta} - c \\ i + j &= k \text{ for given } s_0, h_0 \end{aligned} \tag{8}$$

with appropriate transversality conditions holding.

3.2 Solving the model

By adopting a constant elasticity utility function, the current-value Hamiltonian is

$$H = \frac{c^{1-\xi}}{1-\xi} + \lambda\left(i - \delta s + f(s)\right) + \mu\left(\gamma(k-i)^{\alpha} s^{\alpha+\beta} l^{\beta} h^{1-\alpha-\beta} - c\right). \tag{9}$$

The first order conditions are

$$H_c = c^{-\xi} - \mu = 0, \tag{10}$$

$$H_i = \lambda - \mu\alpha\gamma(k-i)^{\alpha-1} s^{\alpha+\beta} l^{\beta} h^{1-\alpha-\beta} = 0, \tag{11}$$

[4] Note that in parallel to Romer's (1990) variety model where he defined the aggregate capital stock as $K \equiv \eta A x$ with K as capital stock, A as stock of knowledge and x the marginal products of each capital good. We take η equal to one.

$$\frac{\dot{\lambda}}{\lambda} = \rho + \delta - f'(s) - \frac{\mu}{\lambda}(\alpha+\beta)\gamma(k-i)^{\alpha} s^{\alpha+\beta-1} l^{\beta} h^{1-\alpha-\beta}, \tag{12}$$

$$\frac{\dot{\mu}}{\mu} = \rho - (1-\alpha-\beta)\gamma(k-i)^{\alpha} s^{\alpha+\beta} l^{\beta} h^{-\alpha-\beta}, \tag{13}$$

and (8). From (10) and (13), we obtain

$$\frac{\dot{c}}{c} = -\frac{1}{\xi}\left(\rho - (1-\alpha-\beta)\gamma\left(k-i\right)^{\alpha} s^{\alpha+\beta} l^{\beta} h^{-\alpha-\beta}\right). \tag{14}$$

Rearranging (11) gives

$$\frac{\lambda}{\mu} = \alpha\gamma\left(k-i\right)^{\alpha-1} s^{\alpha+\beta} l^{\beta} h^{1-\alpha-\beta}. \tag{15}$$

From (12) and (15),

$$\frac{\dot{\lambda}}{\lambda} = \rho + \delta - f'(s) - \frac{(\alpha+\beta)(k-i)}{\alpha s}. \tag{16}$$

By taking a derivative of log of (15) with respect to t, the equation of motion of i is

$$\dot{i} = \frac{k-i}{\alpha-1}\left\{(\alpha+\beta)\frac{\dot{s}}{s} + (1-\alpha-\beta)\frac{\dot{h}}{h} - \frac{\dot{\lambda}}{\lambda} + \frac{\dot{\mu}}{\mu}\right\}. \tag{17}$$

Thus the dynamic system of our model can be summarized by

$$\begin{aligned}
\dot{s} &= i - \delta s + f(s) \\
\dot{h} &= \gamma\left(k-i\right)^{\alpha} s^{\alpha+\beta} l^{\beta} h^{1-\alpha-\beta} - c \\
\dot{c} &= -\frac{1}{\xi}\left(\rho - (1-\alpha-\beta)\gamma\left(k-i\right)^{\alpha} s^{\alpha+\beta} l^{\beta} h^{-\alpha-\beta}\right) c \\
\dot{i} &= \frac{k-i}{\alpha-1}\left\{(\alpha+\beta)\left(\frac{i}{s} - \delta + \frac{f\left(s\right)}{s}\right)\right. \\
&\quad \left. - \left(1-\alpha-\beta\right)\frac{c}{h} - \delta + f'\left(s\right) + \frac{\left(\alpha+\beta\right)\left(k-i\right)}{\alpha s}\right\}.
\end{aligned} \tag{18}$$

At a steady state, $\dot{s} = \dot{h} = \dot{c} = \dot{i} = 0$ holds and there are possibly multiple steady states.

3.3 Numerical examples

The following specific function may be used for our numerical exercise to describe the aforementioned agglomeration or crowding-in effect:

$$f\left(s\right) = \frac{m s^{\omega}}{n^{\omega} + s^{\omega}} \geq 0 \tag{19}$$

where there are increasing returns ($f'' > 0$) first and then decreasing returns ($f'' < 0$) after a certain reflection point $\bar{s}$, depicted as a s-shape, m determines the upper limit of f ($\lim_{s \to \infty} f = m$), n determines the strength of the agglomeration effect between two boundaries 0 and m ($\partial f / \partial n < 0$), and ω is a positive real parameter.

Next, we set baseline parameters as $m = 5$, $\omega = 3$, $\rho = .03$, $\xi = 2.5$, $k = 1$, $\delta = .05$, $\gamma = 1$, $\alpha = .3$, $\beta = .3$, $l = 1$. We first study the effect of the strength of agglomeration on the number of steady states and on the steady state values.

	c^*, y^* cons-inc	i^* env-inv	s^* env-stock	h^* edu-stock	eigenvalues
n* = *65 (weakly nonlinear)					
SS_1	44.265	0.567	11.970	590.200	{0.107, -0.077, 0.042, -0.012}
n* = *58					
SS_1	45.748	0.574	12.470	609.976	{0.099, -0.069, 0.041, -0.011}
SS_2	102.556	0.782	39.100	1367.42	{0.051, -0.021, 0.015±0.007i}
SS_3	285.122	0.345	62.639	3801.63	{0.065, 0.038, -0.035, -0.008}
n* = *55					
SS_1	46.843	0.579	12.845	624.576	{0.094, -0.064, 0.041, -0.011}
SS_2	88.586	0.767	32.660	1181.14	{0.050, -0.020, 0.015±0.015i}
SS_3	342.393	0.135	65.456	4565.24	{0.073, -0.043, 0.039, -0.009}
n* = *53 (highly nonlinear)					
SS_1	47.891	0.584	13.209	638.552	{0.089, -0.059, 0.041, -0.011}
SS_2	82.040	0.747	28.977	1093.86	{0.049, -0.019, 0.015±0.019i}
SS_3	373.597	0.005	66.624	4981.29	{0.078, -0.048, 0.040, -0.010}

Table 1: Comparative dynamics for different degrees of agglomeration

Table 1 reports the computed steady state values of c, i, s, h, and y, and the eigenvalues at each steady state for different degrees of agglomeration, $n = 65, 58, 55, 53$. As n decreases, there are stronger agglomeration effects. Consumption equals income at a steady state. This is so as human capital by assumption does not depreciate and $\dot{h} = y - c = 0$ must hold at a steady state. This assumption, however, can be easily relaxed by introducing positive depreciation of human capital without changing the implications of the model's results.

For $n = 65$, we find a unique steady state. As the associated eigenvalues indicate, the steady state is a saddle stable point and thus this steady state is actually reached in the long run for any given initial state. For $n = 58, 55, 53$, with stronger agglomeration effects, three steady states emerge. In that case, computing eigenvalues at each steady state confirms that the first and the third steady states are saddle stable points and the middle steady state is an unstable point. This suggests that there are two basins of attraction associated with the two stable steady states and a threshold separating two domains of attraction. We may interpret one domain of attraction as a poverty trap and the other domain as a take-off region. It leads to an open-ended dynamics where a community with the initial human and environmental resources above a certain threshold level tends to reach the upper steady state while a community with these initial resources below the threshold level tends to reach the lower steady state. The results from local dynamic analysis in Table 1, however, does

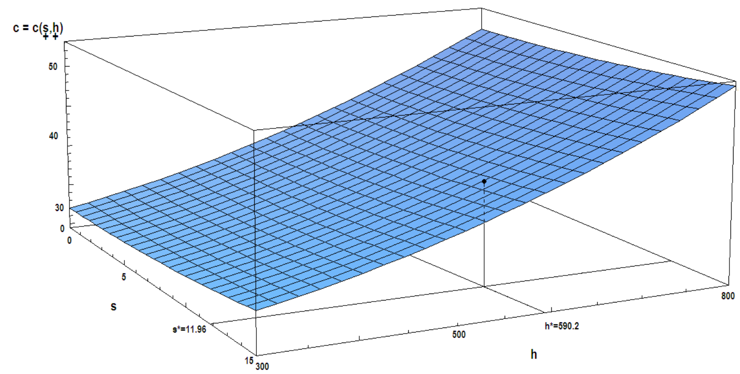

Figure 1: Policy function c for a given state (s, h) for $n = 65$

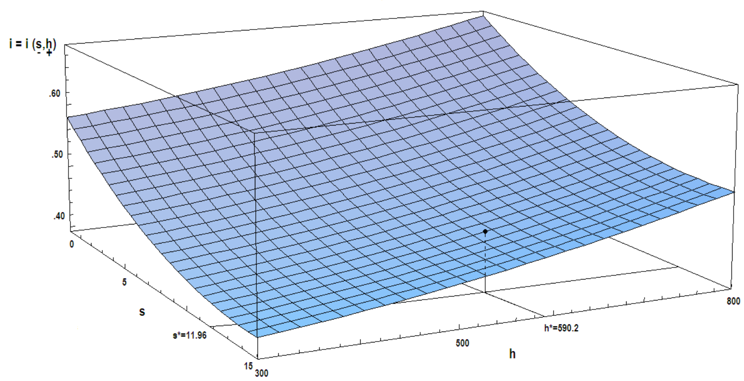

Figure 2: Policy function i for a given state (s, h) for $n = 65$

not allow us to find which steady state is actually reached for a given initial state. Studying this issue requires us to undertake a global dynamic analysis in Subsection 3.5.

3.4 Derivation of policy function

A policy function calculates the best response to a given state by the community's decision board that maximizes the welfare of the community. We use a numerical algorithm to compute the optimal values of the decision variables, c and i, and visualize them in the state space (s, h).[5] Figure 1 and 2 show the optimal consumption function and the optimal social investment function respectively for the example $n = 65$ where

[5]See Appendix 1 of how those two decision variables are numerically computed. We also want to note that the two decision variables, c and i, do not have to be chosen completely optimally. Their magnitude should only approximately be correct for the out of the steady state dynamics.

a unique steady state arises at $s^* = 11.97$ and $h^* = 590.2$. We find, in Figure 1, that consumption should increase when environment improves ($\partial c/\partial s > 0$) and/or human capital increases ($\partial c/\partial h > 0$). Moreover, consumption can grow fastest when environment and human capital formation are compatible. From Figure 2, investments in environment increase when human capital increases ($\partial i/\partial h > 0$) and decrease when environment improves ($\partial i/\partial s < 0$). For a given level of human capital, an environmentally poor community produces and consumes less and spends a larger portion of transfer in social investments to improve environment. On the other hand, an environmentally rich community produces and consumes more and spends less on social investments and more on one-time public services. The quality of environment does not decrease even with little social investments in such a community as stronger agglomeration exists where private investments play a critical role for maintaining the quality of environment.

3.5 Global dynamics

When there are two stable steady states as seen in the examples for $n = 53$, 55, 58, deriving policy functions is not as easy as deriving the policy functions for the example with a unique steady state as in the example for $n = 65$. In that case, finding the optimal paths and the steady state that is actually reached require comparing the present values for a given initial state for the paths to the different stable steady states, namely a global dynamic analysis. Recent technical development in numerical algorithm, however, may help us to analyze the global dynamics. Grüne *et al.* (2005), for example, study a model with one control variable and one state variable, Haunschmied *et al.* (2003) and Grüne and Semmler (2004) study a model with one control variable and two state variables. As our model involves two control variables and two state variables, the dimension is even higher than for these existing studies. The numerical methods to solve this problem have not been developed far enough, but the existing studies tell us that we can make a conjecture on possible scenarios. Three different scenarios are expected to appear from one to the next in the following order as n decreases:

(Sc.1) Dominance of poverty: the optimal path, for any initial given state (s, h), leads to the lower steady state SS_1 as it yields the largest present value to the community.[6] The path to the upper steady state SS_3 will never be chosen in this scenario.

(Sc.2) State-dependent dynamics[7]: the optimal path may lead to the lower steady state SS_1 or to the upper steady state SS_3 depending on the initial state. There exists a threshold line in the state space, beyond which the path to SS_3 yields

[6] Here defined as welfare for the community.

[7] This case was discovered by Skiba (1978).

the largest present value and below which the path to SS_3 yields the largest present value. When the initial state is given at any point on this threshold line, it is indifferent to the community which path is chosen – to SS_1 and to SS_3, as they yields the same present value.

(Sc.3) Dominance of take-off: the optimally controlled path, for any initial given state, leads to SS_3 as it yields the largest present value. The path to the lower steady state SS_1 will never be chosen in this scenario.

3.6 Degree of inequality and policy measures

When there is a unique steady state, all communities, no matter how large the initial dispersion of income is, converge to that unique steady state in the long run. Inequality therefore tends to decrease as time passes. When there are multiple steady states, on the other hand, inequality can persist. Communities move to the high income steady state y_3^* when the amount of their initial human and environmental resources exceeds the threshold level; otherwise they move to the low income steady state y_1^*. Therefore, as long as dispersion of initial resource is sufficiently diffuse, there are communities that realize low income and communities that realize high income. When we measure income inequality by the distance between two stable steady states, $d^* \equiv y_3^* - y_1^*$, we find, from Table 1, $d^* = 0$, 50.169, 52.611, 53.415 for $n = 65$, 58, 55, 53 respectively. This implies that stronger agglomeration is a potential source of larger income inequality in the long run.

	c^*, y^* cons-inc	i^* env-inv	s^* env-stock	h^* edu-stock	eigenvalues
k = 1.05 (+5%)					
SS_1	52.852	0.619	14.324	704.694	{0.084, -0.054, 0.041, -0.011}
SS_2	83.636	0.761	27.657	1115.151	{0.049, -0.019, 0.015±0.020i}
SS_3	387.196	0.006	67.387	5162.617	{0.079, -0.049, 0.040, -0.010}
k = 1.10 (+10%)					
SS_1	58.602	0.657	15.666	781.355	{0.077, -0.047, 0.040, -0.010}
SS_2	84.209	0.771	26.128	1122.791	{0.049, -0.019, 0.015±0.020i}
SS_3	400.526	0.007	68.132	5340.346	{0.080, -0.050, 0.040, 0.010}
k = 1.15 (+15%)					
SS_1	65.947	0.702	17.514	879.294	{0.068, -0.038, 0.038, -0.008}
SS_2	82.974	0.775	24.107	1106.316	{0.050, -0.020, 0.015±0.016i}
SS_3	413.615	0.009	68.859	5514.871	{0.082, -0.052, 0.040, -0.010}
k = 1.20 (+20%)					
SS_1	426.489	0.011	69.573	5686.525	{0.083, -0.053, 0.041, -0.011}

Table 2: Comparative dynamics for different amounts of transfer

Our model shows that persistent and larger inequality possibly arises when strong economies of agglomeration exist. It is true that the model's outcome, even if it shows potentially large inequality, is a result of welfare-maximizing decision by each community; on the other hand the model's result implies that the initial human and environmental resources can be the critical determinants of the long-run outcome for a community.

There are a number of undesirable outcomes that are likely to arise from substantial inequality such as social instability and strong tensions between poor and rich neighborhoods. When these issues are considered, some policies to reduce inequality are advisable. We are interested here, without changing the model, in investigating the effect of the amount of transfer k as a policy parameter on the degree of inequality. Given the assumption that the community's decision board receives the constant amount of transfer every time period that can be used either for social investments i or for spending on public services j, it is easy to predict that an increase in k increases the steady state level of income. It is, however, not so obvious if an increase in k reduces potential inequality d^*. We start from the situation of $n = 53$ in Table 1 where $k = 1$. There are high income y_1^* and low income y_3^* that can be reached by communities in the long run. If no policies are pursued, income inequality $d^* \equiv y_3^* - y_1^* = 53.415$ may be realized. Our simulations focus on the change of global dynamics by varying the transfer k. Table 2 reports the steady state values for different ks. As k increases, not only do steady-state incomes y_1^* and y_3^* increases as we predicted, but inequality also decreases to $d^* = 53.063$, 52.466, 51.346 respectively. Moreover at $k = 1.2$ the lower and middle steady state disappears and a unique steady state arises in the high income region. This result implies that, as much as inequality is concerned, the policy maker should apply the highest amount of transfer to the regions that are affected by inequality most.

4 Poverty trap in the data

The model's result suggests that highly non-linear agglomeration can cause a social lock-in with respect to human capital, environment, income, and consumption across communities. Such lock-in effects can be found in the data. We shall show data that indicate a lock-in in the educational status.

4.1 Data

The data cover math proficiency passage rates of 608 school districts in Ohio during 1990-2002. Ohio is about as representative of the U.S. as any state gets. It has six fairly large urban areas with population between 600,000 and 2.2 million, along with numerous small cities and rural areas. It has prosperous suburban school districts and poverty-stricken inner cities, prosperous farming communities throughout the western and central parts of the state and poor Appalachian areas in the southeast. The uneven prosperity of the state led to a successful challenge of Ohio's school funding formula in the 1990s. In response, the state legislature increased tax revenue devoted to schools, with property-rich school districts getting less state funding and property-poor school districts getting increased state funding. As a result, between

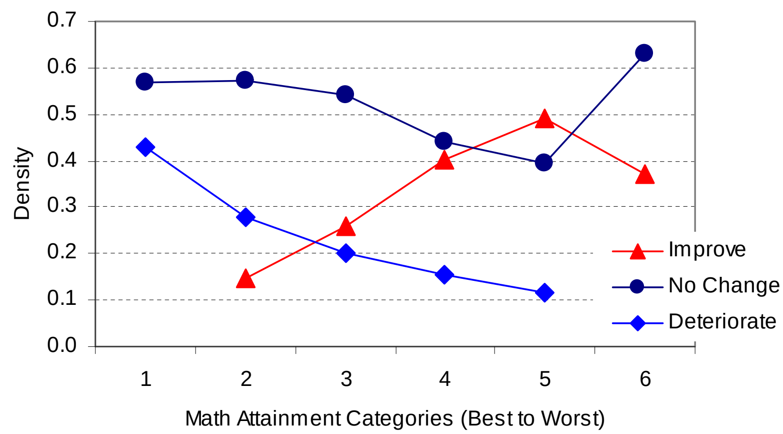

Figure 3: Transition of math attainment

the 1994-1995 school year and the 2005-2006 school year, nominal expenditures per pupil rose from \$8188 to \$13,558 in the relatively prosperous Cincinnati suburban school district of Princeton, and from \$4204 to \$9286 in the poor southeastern school district of Vinton County, representing 27% and 70% increases in real spending.

We first break the percentage passage rates into 6 categories; 1(90s), 2(80s), 3(70s), 4(60s), 5(50s), 6(<50) −1 is the best and 6 is the worst, and then construct ten independent one-year transition matrices.[8] They are reported in Table 3 in Appendix 2. By simply taking the average of ten transition probabilities for each entry, we get a new transition matrix "Avr10". It is a column stochastic matrix where its column sums are unity. The last column reports the resulting ergodic distribution assuming that the Markov chain has stationary transition probabilities (homogeneous chains).[9]

4.2 Inequality in educational attainment

As we only focus on the direction (improve, no change, or deteriorate) of the transition of math attainment, the matrix Avr10 can be simply summarized in Figure 3. We find, for example, that school districts that are in a math-attainment category 3 move within a year to the better categories, i.e., categories 1 or 2, with 25.74% chance, stay in the same category with 54.13% chance, and move to the worse categories, i.e., categories 4, 5, or 6, with 20.13% chance. Though both categories 5 and 6 are low-performance categories, category 5 has the highest chance of take-off to the upper

[8]There is no data for 1994. It is excluded due to the major test procedure change by the Ohio Department of Education. Thus, both 1993-94 and 1994-95 transition matrices cannot be constructed. For the same reason, the transition from 1993 (before the change) to 1995 (after) is inappropriate for inclusion to derive the average transition.

[9]For details, see Appendix 2.

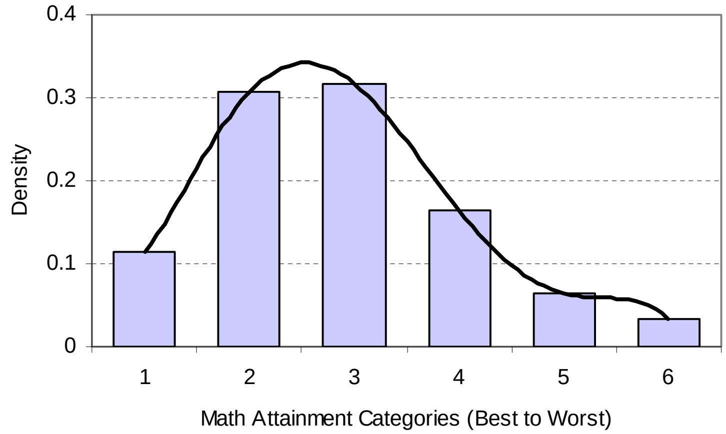

Figure 4: Ergodic distribution of math attainment

categories (49%) and the lowest chance of immobility (39.28%) while category 6 has the highest chance of immobility (62.92%) and the much lower chance of take-off (37.08%). This implies that there is a threshold separating two domains of attraction between categories 5 and 6 and that category 6 can be viewed as an educational-poverty trap.

The ergodic distribution of math attainment reported in Avr 10 is depicted in Figure 4. The ergodic distribution is the unique stationary distribution, the long-run outcome for a given trend. The highly skewed-right distribution therefore indicates persistent inequality in the long run. While most school districts seem to be normally concentrated about categories 2 and 3, there is no tendency that school districts in categories 5 and 6 gradually disappear. This again implies that there may be two attractors and the low-attainment attractor prevents the long-run distribution to be normal.

5 Conclusions

In this paper, we discuss the role of social environment and human capital formation in explaining the presence of inequality that persists. We study a welfare-maximizing problem undertaken by the community's decision board that is concerned with the welfare of a typical household in the community using a Romer (1990) type variety model. We also take the so-called agglomeration (or crowding-in) effect into account in the underlying dynamics of the social environment. With the help of a numerical method, we derive the policy functions that can be a guidance to the community's decision board for choosing the levels of consumption, investment in education and investment in social environment for a given circumstance. We also find that with a stronger agglomeration effect, our model exhibits two attractors. We can interpret one attractor as a poverty trap and the other attractor as a take-off region. The potential inequality, when measured with the distance between the incomes at two stable steady states, enlarges as agglomeration effect gets stronger. Finally, we study a policy that aims at reducing inequality. By investigating the effect of the amount of transfer as a policy parameter on the degree of inequality, we find that applying a larger amount of transfer not only reduces the size of the poverty trap but also reduces the potential inequality. This may suggest that the largest amount of transfer should be applied to the region with the largest inequality.

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Appendix 1: Numerical solution method

We here briefly describe the dynamic programming algorithm as applied in Grüne and Semmler (2004) that enables us to numerically solve the dynamic model as proposed in Section 4. The feature of the dynamic programming algorithm is an adaptive discretization of the state space which leads to high numerical accuracy with moderate use of memory.

Such algorithm is applied to discounted infinite horizon optimal control problems of the type introduced in Section 4. In our model variants we have to numerically compute $V(x)$ for

$$V(x) = \max_u \int_0^\infty e^{-r} f(x,u)dt$$

$$s.t. \dot{x} = g(x,u)$$

where u represents the control variable and x a vector of state variables.

In the first step, the continuous time optimal control problem has to be replaced by a first order discrete time approximation given by

$$V_h(x) = \max_j J_h(x,u), \quad J_h(x,u) = h \sum_{i=0}^{\infty} (1-\theta h) U f(x_h(i), u_i) \tag{A1}$$

where x_u is defined by the discrete dynamics

$$x_h(0) = x, \ x_h(i+1) = x_h(i) + hg(x_i, u_i) \tag{A2}$$

and $h > 0$ is the discretization time step. Note that $j = (j_i)_{i \in \mathbb{N}_0}$ here denotes a discrete control sequence.

The optimal value function is the unique solution of a discrete Hamilton-Jacobi-Bellman equation such as

$$V_h(x) = \max_j \{hf(x,u_o) + (1+\theta h)V_h(x_h(1))\} \tag{A3}$$

where $x_h(1)$ denotes the discrete solution corresponding to the control and initial value x after one time step h. Abbreviating

$$T_h(V_h)(x) = \max_j \{hf(x,u_o) + (1-\theta h)V_h(x_h(1))\}, \tag{A4}$$

the second step of the algorithm now approximates the solution on grid Γ covering a compact subset of the state space, i.e. a compact interval $[0,K]$ in our setup. Denoting the nodes of Γ by $x^i, i = 1, ..., P$, we are now looking for an approximation V_h^Γ satisfying

$$V_h^\Gamma(X^i) = T_h(V_h^\Gamma)(X^i) \tag{A5}$$

for each node x^i of the grid, where the value of V_h^Γ for points x which are not grid points (these are needed for the evaluation of T_h) is determined by linear interpolation.

We refer to the paper cited above for the description of iterative methods for the solution of (A5). Note that an approximately optimal control law (in feedback form for the discrete dynamics) can be obtained from this approximation by taking the value $j^*(x) = j$ for j realizing the maximum in (A3), where V_h is replaced by V_h^Γ. This procedure in particular allows the numerical computation of approximately optimal trajectories.

In order the distribute the nodes of the grid efficiently, we make use of a posteriori error estimation. For each cell C_l of the grid Γ we compute

$$\eta_l := \max_{k \in c_l} \mid T_h(V_h^\Gamma)(k) - V_h^\Gamma(k) \mid .$$

More precisely we approximate this value by evaluating the right hand side in a number of test points. It can be shown that the error estimators η_l give upper and lower bounds for the real error (i.e., the difference between V_j and V_h^Γ) and hence serve as an indicator for a possible local refinement of the grid Γ. It should be noted that this adaptive refinement of the grid is very effective for computing steep value functions and models with multiple equilibria, see Grüne and Semmler (2004).

Appendix 2: Derivation of ergodic distribution

Our study is based on 14 years of data (1990-2002) of math proficiency passage rates of 608 school districts in Ohio. We break the percentage passage rates into 6 categories; 1(90s), 2(80s), 3(70s), 4(60s), 5(50s), 6(<50) so that for each of 14 years the number of school districts falls into each of the six categories. Category 1 is the best and category 6 is the worst. Note that there is no data for 1994. It is excluded due to the major test procedure change by the Ohio Department of Education. In 1994, Ohio started letting students take the 9th grade proficiency test in 8th grade. So the numbers reported for 9th grade proficiency include the 8th graders who passed. Based on the data, we can construct 11 independent one-year transition matrices such as 1990-91, 91-92, ..., 2001-02. Those are reported in Table 3. Each matrix is a column stochastic matrix where its column sums are unity. Next, by simply taking the average transition probabilities, we obtain the averaged transition matrix for the 11 transition matrices. This is reported as Avr11 in Table 3. There is however a concern with the transition matrix 1993-95 that crosses over the 1994 major change of the test procedure. The transition can be largely affected by the change. Thus, we exclude the matrix and obtain a new average transition matrix for the 10 transition matrices as Avr10.

The percentage ergodic distribution of 608 school districts is calculated in the last column of each transition matrix assuming that the Markov chain has stationary transition probabilities (homogeneous chains). Table 3 reports the ergodic distributions. The ergodic distribution is the unique stationary distribution and useful to estimate a long-run outcome based on the recent trend in the school performance.

We find that most distributions are highly skewed right. Only a few school districts are left in categories 5 and 6 while most districts are in categories 2 and 3. Note that the ergodic distribution for the 1993-95 transition matrix, as we expected, seems affected by the major procedure change in 1994. Therefore, it should be better to use Avr 10 than Avr 11.

Let's P be the irreducible transition matrix. Then the postmultiplication by **1** the vector with unity in each position gives

$$\mathbf{1}'P = \mathbf{1}'$$

by stochasticity of P where 1 is an eigenvalue and **1** is a corresponding left eigenvector.

Since all column sums of P are equal and the Perron-Frobenius eigenvalue lies between the largest and the smallest

$$\min_j \sum_{i=1}^{n} p_{ij} \leq r \leq \max_j \sum_{i=1}^{n} p_{ij}$$

where $r \geq |\lambda|$ for any eigenvalue λ of P, 1 is the Perron-Frobenius eigenvalue of P and **1** is the corresponding left Perron-Frobenius eigenvector.

Let's define the corresponding right eigenvector as a column vector v that is normed as

$$\mathbf{1}'v = 1.$$

Then, we have

$$Pv = v$$

where v is the vector of probability distribution.

Theorem 1 *An irreducible Markov chain has a unique stationary distribution given by the solution v of $Pv = v$, $\mathbf{1}'v = 1$.*

Proof. Any initial probability distribution Π_0 is called a stationary distribution if

$$\Pi_0 = \Pi_k,\ k = 1, 2, ...$$

If Π_0 is a stationary distribution,

$$P\Pi_0 = \Pi_0,\ \Pi_0 \geq 0,\ \mathbf{1}'\Pi_0 = 1.$$

By uniqueness of the right Perron-Frobenius eigenvalue of P, $\Pi_0 = v$. ■

Theorem 2 *(Ergodic Theorem for primitive Markov chains) As $k \to \infty$, for a primitive Markov chain, $P^k \to v\mathbf{1}'$ elementwise where v is the unique stationary distribution of the Markov chain and the rate of approach to the limit is geometric.*

Proof. See Seneta (2006) Theorem 1.2, p. 9. ■

Corollary 3 *The unique stationary distribution is independent from the initial distribution.*

Proof. For any initial probability distribution Π_0, as $k \to \infty$

$$P^k \Pi_0 \to v \mathbf{1}' \Pi_0.$$

Since $\mathbf{1}'\Pi_0 = 1$,

$$P^k \Pi_0 \to v.$$

■

We used Mathematica to obtain the ergodic distributions reported in Table 3. For each 6×6 transition matrix in Table 3, we first compute the eigenvalues, then find the corresponding right eigenvector to the Perron-Frobenius eigenvalue 1, and normalize it so that $\mathbf{1}'v = 1$ By using the following steps, the same results should be reproduced:

Step 1: Specify the transition matrix P

In[1]:= P = {{p11, p12, ..., p16}, {p21, p22, ..., p26},...{p61, p62, ..., p66}}; MatrixForm[P]

Step 2: Obtain eigenvalues

In[2]:= Eigenvalues[P]

Mathematica sorts eigenvalues, if they are numeric, in order of decreasing absolute value. Since the Perron-Frobenius eigenvalue of P is $r \geq |\lambda|$ for any eigenvalue λ of P, the first eigenvalue should be 1.

Step 3: Obtain eigenvectors

In[3]:= MatrixForm[Eigenvectors[P]]

Mathematica returns the matrix of eigenvectors. The corresponding Perron-Frobenius right eigenvector to the first eigenvalue 1 should be the first row.

Step 4: Normalize the eigenvector so that $\mathbf{1}'v = 1$.

The obtained v is the stationary distribution reported in Table 3.

91\90	1	2	3	4	5	6	Ergodic dist
1	0.5	0.2667	0.0244	0	0	0	0.2154
2	0.5	0.5333	0.3171	0.0822	0.0068	0	0.3857
3	0	0.1333	0.5122	0.3425	0.1293	0.0121	0.1979
4	0	0.0667	0.122	0.3699	0.3401	0.1273	0.1127
5	0	0	0	0.137	0.4082	0.2879	0.0465
6	0	0	0.0244	0.0685	0.1156	0.5727	0.0419

92\91	1	2	3	4	5	6	Ergodic dist
1	0.5	0.1034	0	0	0	0	0.0388
2	0.3333	0.5517	0.1972	0.04	0.0061	0.0047	0.1873
3	0.1667	0.3448	0.5211	0.264	0.0848	0.0377	0.3074
4	0	0	0.1972	0.48	0.3152	0.1226	0.2278
5	0	0	0.0704	0.176	0.4	0.3019	0.1484
6	0	0	0.0141	0.04	0.1939	0.533	0.0904

93\92	1	2	3	4	5	6	Ergodic dist
1	0.3333	0.1538	0.0194	0	0	0	0.0453
2	0.6667	0.4872	0.2136	0.0263	0.0064	0	0.168
3	0	0.2564	0.4757	0.2105	0.0764	0.0265	0.2233
4	0	0.1026	0.2136	0.5	0.2994	0.1258	0.2695
5	0	0	0.068	0.2368	0.414	0.3046	0.1892
6	0	0	0.0097	0.0263	0.2038	0.543	0.1047

96\95	1	2	3	4	5	6	Ergodic dist
1	0.5294	0.125	0.0053	0	0	0	0.0667
2	0.4412	0.5703	0.1958	0.0385	0	0.027	0.2376
3	0.0294	0.2578	0.5556	0.2885	0.125	0	0.3194
4	0	0.0469	0.2116	0.4744	0.375	0.1081	0.2301
5	0	0	0.0317	0.1667	0.4063	0.2162	0.0989
6	0	0	0	0.0321	0.0938	0.6486	0.0474

97\96	1	2	3	4	5	6	Ergodic dist
1	0.6	0.1288	0.0208	0.0068	0	0	0.0827
2	0.3714	0.5379	0.1458	0.0473	0.0152	0	0.1936
3	0.0286	0.2727	0.599	0.3041	0.0303	0.0286	0.319
4	0	0.053	0.2083	0.4797	0.3182	0.0286	0.2215
5	0	0.0076	0.026	0.1486	0.5606	0.1143	0.1149
6	0	0	0	0.0135	0.0758	0.8286	0.0683

98\97	1	2	3	4	5	6	Ergodic dist
1	0.6279	0.1417	0.0101	0	0	0	0.1768
2	0.3256	0.675	0.2864	0.0709	0	0	0.4441
3	0.0465	0.175	0.5829	0.4397	0.1014	0.0278	0.2854
4	0	0.0083	0.1156	0.3546	0.4928	0.0833	0.0709
5	0	0	0.005	0.1206	0.3623	0.2222	0.0175
6	0	0	0	0.0142	0.0435	0.6667	0.0053

99\98	1	2	3	4	5	6	Ergodic dist
1	0.6739	0.1049	0.0048	0.009	0	0	0.1049
2	0.2826	0.6235	0.244	0.018	0	0	0.2936
3	0.0435	0.2407	0.5311	0.3063	0.1569	0.0345	0.3162
4	0	0.0309	0.201	0.5225	0.4314	0.1724	0.2085
5	0	0	0.0144	0.1351	0.2745	0.2414	0.053
6	0	0	0.0048	0.009	0.1373	0.5517	0.0238

00\99	1	2	3	4	5	6	Ergodic dist
1	0.6	0.1317	0	0.0076	0.0256	0	0.0897
2	0.34	0.503	0.2051	0.0379	0	0	0.2551
3	0.06	0.3293	0.641	0.4091	0.0769	0.08	0.4403
4	0	0.0299	0.1385	0.4394	0.4359	0.04	0.1569
5	0	0.006	0.0154	0.1061	0.3077	0.32	0.043
6	0	0	0	0	0.1538	0.56	0.015

01\00	1	2	3	4	5	6	Ergodic dist
1	0.6296	0.1849	0.0289	0	0	0	0.2438
2	0.3519	0.6438	0.3058	0.0833	0.0526	0	0.4526
3	0.0185	0.1712	0.5041	0.4815	0.1316	0.05	0.2279
4	0	0	0.1446	0.3796	0.3684	0	0.0607
5	0	0	0.0165	0.0556	0.3947	0.25	0.0127
6	0	0	0	0	0.0526	0.7	0.0022

02\01	1	2	3	4	5	6	Ergodic dist
1	0.7059	0.1162	0	0	0	0	0.155
2	0.2206	0.6162	0.3495	0.0556	0	0	0.3924
3	0.0735	0.2475	0.4903	0.4667	0.1333	0	0.316
4	0	0.0202	0.1553	0.4	0.3667	0.0625	0.1074
5	0	0	0	0.0778	0.4	0.25	0.0184
6	0	0	0.0049	0	0.1	0.6875	0.0108

Avr10	1	2	3	4	5	6	Ergodic dist
1	0.57	0.1457	0.0114	0.0023	0.0026	0	0.1133
2	0.3833	0.5742	0.246	0.05	0.0087	0.0032	0.306
3	0.0467	0.2429	0.5413	0.3513	0.1046	0.0297	0.3167
4	0	0.0358	0.1708	0.44	0.3743	0.0871	0.1648
5	0	0.0014	0.0247	0.136	0.3928	0.2509	0.0647
6	0	0	0.0058	0.0204	0.117	0.6292	0.0344

Table 3: Markov transition matrices for 1990-2001 Ohio